A GOAL-CONSTRAINT FORMULATION FOR MULTI-ITEM INVENTORY SYSTEMS

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THESIS

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ABSTRACT

Historically multi-item inventory control has been modeled by assuming that each item can be treated independently in a variable cost minimization formulation. In this paper independence between items is not assumed. Constraints on total system operating characteristics create inter-item dependencies. Optimal policies are determined from a goal-constraint formulation. This is done without reliance upon unknown parameters such as order cost and carrying cost which the traditional theory leans on heavily. A group of models are presented, with necessary and sufficient conditions for optimal solutions provided for each. In addition, solution algorithms are indicated for the major models. An algorithm for verification of sufficiency conditions for a non-convex objective function is also provided.

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CHAPTER I

INTRODUCTION

1.1 BACKGROUND

The great majority of research in the field of inventory theory has been associated with the treatment of a single item. The object of such research is to determine decision rules which tell us how much to order and when to order to minimize variable costs. In order to implement these decision rules one must know, or at least be able to estimate the order cost, holding cost, and shortage cost. Another implicit assumption is that investment capital is available to buy and hold the optimal order quantity and safety level.

In the military the situation is somewhat different. The cost parameters mentioned previously are difficult, if not impossible, to estimate. Furthermore, they may not always be meaningful. An example of this is the inventory carrying cost. A major portion of the carrying cost is the opportunity cost related to alternative uses of the investment in stock. Within the Navy, the budget funds allocated to stock cannot be used for any other purpose. In addition to this problem, each military inventory is under severe budget restrictions.

In order to understand these restrictions, some discussion of the budget process is necessary. The manager of a Naval Supply activity receives funds from two sources. The first source of funds is referred to as Operations and Maintenance funds. As the name suggests, these funds support the operations of the activity which include salaries, providing for maintenance of buildings, purchase of minor equipments, and replenishment of material necessary to the operation of the activity. These funds cannot be used for the acquisition of items for general stock. The second budget source provides funds for acquisition of items for general stock. This budget can be in terms of either a limit on average investment or a procurement limitation. Moreover, these resources cannot be transferred to the Operations and Maintenance budget. Thus one finds within the military existing constraints on funds available to purchase stock or those available to initiate and process orders and receipts.

These differences between the military situation and the assumptions made in classical theory have suggested some of the questions to be explored in this study.

These questions may be stated in a more precise form as follows:

1. Is the single item theory sufficient to provide decision rules for the multi-item inventories in the Navy?

2. Is the assumption that the order cost, holding cost, and shortage cost are known parameters reasonable or even necessary?

3. Does the addition of budget constraints to the problem require a different basic model?

It is proposed that a more adequate approach is that of a multi-item inventory model based upon an investment limit and an order constraint.

Depending upon the situation, the military inventory manager attempts either to minimize shortages or to minimize weighted shortages. In the first case the tidewater or first line stock point wants to minimize the number of times a customer is turned away due to a lack of material at that activity. In the second instance, weighted shortages are minimized because some projects are more essential from a military standpoint than others. Thus, for the military, the cost of being out of an essential part is greater than the cost of being out of an ordinary part.

The above arguments have motivated the construction of a multi-item inventory model which attempts to satisfy one of the above goals subject to a constraint on average investment as well as a constraint on the number of orders per year.

In order to have other than a trivial solution, every inventory model must have some built-in control to prevent either an infinite order quantity or, on the other extreme, an infinite number of orders. For instance, in the Economic Order Quantity inventory model the order cost prevents ordering after each demand while the holding cost prevents an infinite amount of inventory on hand. The optimal solution then achieves a trade-off between the order cost and the holding cost.

Now we observe that a constraint on the average inventory would achieve the same purpose as the holding cost and a constraint on number of orders would achieve the same purpose as the order cost. Moreover, the budget constraint and the order constraint are more realistic since they are relatively easy to obtain and the inventory manager, who cannot be expected to understand the mathematical theory intimately, has a better intuitive feeling for these constraints than he would have for the order cost and holding cost parameters.

1.2 CURRENT NAVY PROGRAMS

Currently the Navy has two programs which provide decision rules for stock points and inventory control points.

The first program which governs the management of retail material of Navy stock points is known as the Variable Operating and Safety Level (VOSL) program. The VOSL program is a model of the following situation. The stock point receives an Operations and Maintenance allotment which, among other things, fixes the personnel ceiling at that activity. This in some sense limits the maximum number of orders which can be written. In addition, the stock point receives an investment limit which cannot be exceeded. The stock point manager then strives to minimize some function of shortages while remaining within the above constraints. In the case of the stock point we have essentially a single warehouse system.

The second program which governs the purchase of Navy wholesale material is known as the Uniform Inventory Control Point program. However, in the case of the inventory control point, we have a multi-echelon, multi-warehouse system. On the ICP level, the inventory manager is given both a procurement and personnel budget. In this instance, the manager again attempts to minimize shortages within the multi-echelon framework. On the stock point level a similar situation exists with the exception that the inventory manager attempts to minimize some function of shortages with respect to his single warehouse system.

In each program the manager attempts to minimize some function of shortages while subject to some constraint on investment and a constraint on orders. This observation leads to the formulation of the models discussed in Chapter 2.

CHAPTER 2

PROBLEM FORMULATION

2.1 BASIC FORMULATION OF THE PROBLEM

As suggested by the introduction, some other formulation of the inventory problem seems desirable. Such a formulation would not be based on minimizing variable costs in the classic sense, but rather on minimizing some more reasonable objective. For this study the objective will usually be some function of weighted units short.

In general, the basic problem can be stated as

$$\min f(X,Y)$$

subject to:

$$g_j(X,Y,\Lambda) \leq 0 \quad j = 1,1,\ldots,m \ .$$

where X and Y represent vectors of decision variables and Λ represents a vector of known parameters. Throughout this paper we will indicate vectors by non-subscripted capital letters. The objective function $f(X,Y)$ can be expected units short, weighted units short or some other function of shortages. The constraints can represent limitations on the expected number of orders and restrictions on average investment or a procurement budget.

2.2 CONTINUOUS REVIEW FORMULATION

If one is given an inventory of n items, an average investment limit, and a limit on the number of orders then the following continuous review problem can be formulated.

2.2.1 Constraint Formulation

Hadley and Whitin [1] show that under continuous review the average on hand quantity (m_i) for the ith item is

$$r_i + \frac{Q_i}{2} - \mu_i = m_i \ , \tag{2.1}$$

where

r_i = the reorder point for the ith item,

Q_i = the order quantity for the ith item,

$\mu_i = \int_0^\infty x \ f_i(x,\tau_i)dx,$

$f_i(x,\tau_i)$ is the density of lead time demand for the ith item, and r_i and Q_i ($i = 1,2,\cdots n$) are the decision variables.

To convert (2.1) to average on hand investment per item we multiply by the price of the ith item (c_i):

$$c_i(r_i + \frac{Q_i}{2} - \mu_i) = c_i m_i \ . \tag{2.2}$$

Summing over the entire inventory, the average on hand investment would be

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} - \mu_i) = \sum_{i=1}^{n} c_i m_i \ . \tag{2.3}$$

Hadley and Whitin [1] also show the average number of orders (d_i) per unit time for the ith item in steady state is

$$\frac{\lambda_i}{Q_i} = d_i \ , \tag{2.4}$$

where

$$\lambda_i = \int_0^\infty x \ f_i(x)dx \quad \text{and} \quad f_i(x)$$

is the density of demand per unit time. Then the average number of orders per unit time for the entire inventory would be

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} = \sum_{i=1}^{n} d_i \quad . \tag{2.5}$$

Therefore the constraints on average investment and on the number of orders can be written as

$$\sum_{i=1}^{n} c_i (r_i + \frac{Q_i}{2} - \mu_i) \leq K_1 \tag{2.6}$$

and

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \leq K_2 \quad . \tag{2.7}$$

2.2.2 The Objective Function

Several possible objectives can be proposed. At this point, however, we will select as the objective function the minimization of expected units short per unit time. The solution technique would be similar for any choice of objective function.

From Hadley and Whitin [1] we see the expected shortages per procurement cycle for the ith item can be expressed as

$$\int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx \quad . \tag{2.8}$$

The expected shortages per unit time for the ith them become

$$\frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx \quad . \tag{2.9}$$

The objective can now be stated as the minimization of the total number of shortages per unit time for the entire inventory. The objective function becomes

$$Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx \quad .$$

The basic continuous review formulation can now be stated as

$$\min Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx \ ,$$

subject to

$$\sum_{i=1}^{n} c_i\left(r_i + \frac{Q_i}{2} - \mu_i\right) \leq K_1 \ ,$$

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \leq K_2 \ ,$$

$Q_i \geq 0$, and r_i unrestricted.

2.3 PERIODIC REVIEW FORMULATION

If we are again given an n item inventory and given a time period T between reviews and a limitation on average investment, the periodic review model can be formulated which minimizes expected units short per unit time.

2.3.1 Constraint Formulation

From Hadley and Whitin [1] we see the average inventory for the ith item can be expressed as

$$R_i + \mu_i + \frac{\lambda_i T}{2} = b_i \ , \tag{2.10}$$

where

R_i = the "order-up-to" level of the ith item,

$\mu_i = \int_0^{\infty} x h_i(x,\tau_i+T)\,dx$ where $h_i(x,\tau_i+T)$ is the demand distribution over a lead time plus a review period,

λ_i = mean demand per unit time for the ith item, and

T = the length of time between reviews.

If we assume T is fixed and the same for all items, the $R_i (i=1,2,\cdots,n)$ are the only decision variables.

The average investment for the entire inventory then becomes

$$\sum_{i=1}^{n} c_i \left(R_i - \mu_i - \frac{\lambda_i T}{2}\right) = \sum_{i=1}^{n} c_i b_i \ . \tag{2.11}$$

2.3.2 The Objective Function

From Hadley and Whitin [1] the average number of back orders incurred per unit time is

$$Z_i = \frac{1}{T} \int_{R_i}^{\infty} (x-R_i) h_i(x,\tau_i+T) dx \ . \tag{2.12}$$

Therefore, the periodic review problem given a fixed review period may be stated as

$$Z = \sum_{i=1}^{n} \frac{1}{T} \int_{R_i}^{\infty} (x-R_i) h_i(x,\tau_i+T) dx$$

subject to:

$$\sum_{i=1}^{n} c_i \left(R_i - \mu_i - \frac{\lambda_i T}{2}\right) \leq K_1 \ .$$

CHAPTER 3

SOLUTIONS

3.1 SOLUTION TECHNIQUE

The technique we will use to solve the problems formulated in Chapter 2 is known at the Kuhn - Tucker Theorem. This theorem is often used to solve constrained minimization problems where the constraints are inequalities and/or equalities.

First, the Kuhn - Tucker theorem [2] will be stated as it applies to these problems. Consider the problem:

$$\text{minimize } f(X,Y)$$

subject to:

$$g_i(X,Y) \le 0, \; i=1,\cdots,n \; ,$$

$$g_i(X,Y) = 0, \; i=n+1,\cdots,m \; ,$$

$$y_i \ge 0, \text{ and } x_i \text{ unrestricted.}$$

<u>Theorem 1</u>. If the constraint qualifications are satisfied for the minimization problem, then for (X^o,Y^o) to be an optimal solution it is necessary that (X^o,Y^o) and some ¶ satisfy conditions (1) and (2) for

$$F(X,Y,¶) = f(X,Y) - \sum_{i=1}^{m} \eta_i g_i(X,Y)$$

and

$$\eta_i \le 0, \; i=1,\cdots n,$$

η_i unrestricted for $i=n+1,\cdots,m$.

(1) a. $F_x(X^o,Y^o,¶^o) = 0$

b. $F_y(X_, Y^o,¶^o) \ge 0$ and $Y^o F_y(X^o,Y^o,¶^o) = 0$

(2) a. $F_{\eta_i}(X^o,Y^o,\P^o) \leq 0$ for $i=1,\cdots n$ and $\eta_i \leq 0$

$$\eta_o^T F_{\eta_i}(X^o,Y^o,\P^o) = 0$$

b. $F_{\eta_i}(X^o,Y^o,\P) = 0$ for $i=n+1,\cdots m$ η_i unrestricted.

Notice the theorem does not guarantee a solution. Instead, it says if we can find a solution satisfying conditions (1) and (2) then (X^o,Y^o) satisfy the necessary conditions for a minimum.

3.2 SOLUTION TO A SIMPLIFIED CONTINUOUS REVIEW MODEL

Before solving the general continuous review model stated in section 2.2, we shall consider a less complicated model, a form of which is used in Navy inventory management today. This program was referred to in the introduction as the VOSL program.

Suppose the order quantities are fixed by some other criterion. Specifically we assume

1. Order quantities are determined from the economic order cost equation

$$Q_i = \sqrt{\frac{2A\lambda_i}{Ic_i}} ,$$

where A is the order cost and I is the holding cost, which we assume are the same for all items.

2. $\displaystyle\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} = K_2$.

As stated in section 2.1, the general continuous review problem is

$$\min \quad Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx \qquad (3.1)$$

subject to

$$\sum_{i=1}^{n} c_i (r_i + \frac{Q_i}{2} - \mu_i) \leq K_1 \tag{3.2}$$

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \leq K_2 \quad . \tag{3.3}$$

Immediately we see that assumption 1 fixes the value of Q_i and therefore will eliminate the second constraint in the general problem. While we do not actually know the ratio of A to I, this ratio is implied by the constraint on the number of orders. Assumption (1) implies that

$$Q_i = \frac{1}{K} \sqrt{\frac{\lambda_i}{c_i}} \quad , \tag{3.4}$$

where

$$\frac{1}{K} = \sqrt{\frac{2A}{I}} \quad .$$

However, since the quantity $\frac{\lambda_i}{Q_i}$ is required in equation (3.3) we write (3.4) as

$$\frac{\lambda_i}{Q_i} = K\sqrt{c_i \lambda_i} \quad . \tag{3.5}$$

Substituting (3.5) in equation (3.3) yields

$$\sum_{i=1}^{n} K\sqrt{c_i \lambda_i} = K_2$$

or

$$K = \frac{K_2}{\sum_{i=1}^{n} \sqrt{c_i \lambda_i}} \quad . \tag{3.6}$$

The determination of K then fixes the order quantities from equation (3.4) and eliminates one set of decision variables from the problem. If we substitute (3.4) into equation (3.2), we obtain

$$\sum_{i=1}^{n} c_i(r_i + \frac{1}{2K}\sqrt{\frac{\lambda_i}{c_i}} - \mu_i) \leq K_1 \quad . \qquad (3.7)$$

We can reduce (3.7) to the form

$$\sum_{i=1}^{n} c_i r_i \leq K_1 - \frac{1}{2K}\sum_{i=1}^{n}\sqrt{\lambda_i c_i} + \sum_{i=1}^{n} c_i \mu_i = K_1' . \qquad (3.8)$$

Our simplified problem can now be stated as

$$\min \quad Z = \sum_{i=1}^{n} K\sqrt{\lambda_i c_i}\int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx$$

subject to:

$$\sum_{i=1}^{n} c_i r_i \leq K_1' \quad .$$

3.2.1 Necessary Conditions

To solve the simplified problem we apply theorem 1 from section 3.1. From condition (1) we have

$$-K\sqrt{c_i \lambda_i}\int_{r_i}^{\infty} f_i(x,\tau_i)\,dx - \eta_1 c_i = 0 \ . \qquad (3.9)$$

From condition (2) we have

$$\sum_{i=1}^{n} c_i r_i - K_1' \leq 0 \qquad (3.10)$$

and

$$\eta_1 \left(\sum_{i=1}^{n} c_i r_i - K_1'\right) = 0 \quad . \qquad (3.11)$$

Observe that equation (3.9) can be solved for η_1 yielding

$$\eta_1 = \frac{-K\sqrt{\lambda_i c_i}}{c_i} F_i^{\,c}(r_i) \qquad (3.12)$$

where

$$F_i^c(r_i) \text{ is defined as } \int_{r_i}^{\infty} f_i(x,\tau_i)\,dx.$$

The right-hand side of equation (3.12) is always less than zero unless $F^c(r_i) = 0$. The case where $F^c(r_i) = 0$ will occur if r_i exceeds the largest lead time demand. If we assume this is not the case, (3.10) and (3.11) reduce to

$$\sum_{i=1}^{n} c_i r_i = K_1' \,. \tag{3.13}$$

Thus the conditions for solution to the problem are:

$$\eta_1 = \frac{-K\sqrt{\lambda_i c_i}}{c_i} F_i^c(r_i) \,, \tag{3.14}$$

and

$$\sum_{i=1}^{n} c_i r_i = K_1' \,. \tag{3.15}$$

It is possible to obtain a closed form solution which satisfies equations (3.14) and (3.15). However, generally we cannot solve these equations in closed form unless

$$\int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx$$

exists in closed form and is a relatively simple function of r_i.

Instead let us look for a general method of solving equations (3.14) and (3.15). If we consider equation (3.14), which is

$$\eta_1 = -K\sqrt{\frac{\lambda_i}{c_i}}\, F_i^c(r_i) \,, \tag{3.14}$$

we notice the right hand side has an upper bound of 0 since

$$F_i^c(r_i) \geq 0$$

and a lower bound of

$$-K\sqrt{\frac{\lambda_i}{c_i}}$$

for each i since

$$F_i^C(r_i) \leq 1 \quad .$$

This also implies

$$\eta_1 \geq K\min\left(\sqrt{\frac{\lambda_i}{c_i}}, \sqrt{\frac{\lambda_2}{c_2}}, \cdots, \sqrt{\frac{\lambda_n}{c_n}}\right) \quad .$$

Suppose $\delta = K\min\left(\sqrt{\frac{\lambda_i}{c_i}}\right)$ for all i. Then a suggested solution procedure would be to begin at $\eta_1 = \frac{\delta}{2}$, solve equation (3.14) for the vector r and compute the value of the constraint using equation (3.16), which is

$$\sum_{i=1}^{n} c_i r_i = H \quad . \tag{3.16}$$

If $H > K_1'$ decrease η_1 by $\frac{\delta}{4}$. If $H < K_1'$, increase η_1 by $\frac{\delta}{4}$. Compute the value of the constraint using equation (3.16). If the decrease (or increase) of η_1 has not caused the sense of the inequality to switch, decrease (or increase) η_1 by the same amount $\frac{\delta}{4}$. If the sense of the inequality has changed then reduce the increment to $\frac{\delta}{8}$ and decrease (or increase) η_1, solving for the vector r at each value of η_1 and computing the value of H from (3.16) until the sense of the inequality switches again. Continue until $H = K_1'$ or until H is within some acceptable region of K_1'. This method will converge to the optimal solution rapidly.

This approach is feasible on a high speed computer and takes little time for a large inventory. The limitation of this method is solving for r_i from the equation

$$F_i^c(r_i) = \frac{\eta_1}{K} \sqrt{\frac{c_i}{\lambda_i}} \quad .$$

However, approximations are available for some distributions which cannot be solved in closed form.

3.2.2 Sufficient Conditions

From the Kuhn - Tucker paper [2], if we have a convex objective function and a convex constraint region, the necessary conditions are also sufficient. The condition for applying the Kuhn - Tucker theorem is that the constraint region be convex. Since the constraint under consideration is linear in r, the region is convex. To show $Z(r_i)$ is convex let us consider the equation of the expected units short per unit time,

$$Z_i = K\sqrt{\lambda_i c_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\, dx \; .$$

Now if $\frac{\partial^2 Z_i}{\partial r_i^2} \geq 0$ for all r_i, then Z_i is convex. Taking partial derivatives, we find

$$\frac{\partial Z_i}{\partial r_i} = -K\sqrt{\lambda_i c_i} \int_{r_i}^{\infty} f_i(x,\tau_i)\, dx$$

and

$$\frac{\partial^2 Z_i}{\partial r_i^2} = K\sqrt{\lambda_i c_i}\; f_i(r_i,\tau_i) \quad . \tag{3.17}$$

Equation (3.17) will always be greater than or equal to zero. Under these conditions Z_i is convex. Thus it follows that Z is convex since it is the sum of convex functions.

3.3 PERIODIC REVIEW, FIXED PERIOD SOLUTION

In this section we shall consider the following problem as stated in section 2.3:

$$\min Z = \sum_{i=1}^{n} \frac{1}{T} \int_{R_i}^{\infty} (x-R_i) h_i(x,\tau_i+T)\,dx$$

subject to:

$$\sum_{i=1}^{n} c_i(R_i - \mu_i - \frac{\lambda_i T}{2}) \le K_1 \quad .$$

3.3.1 Necessary Conditions

Applying Theorem 1 we obtain the following equations which will yield the necessary conditions for optimality:

$$-H_i^c(R_i) = \eta c_i T \qquad (3.18)$$

and

$$\sum_{i=1}^{n} c_i(R_i - \mu_i - \frac{\lambda_i T}{2}) = K_1 . \qquad (3.19)$$

The solution of equations (3.18) and (3.19) can be found using the method described in the previous section. If the solution vector R^o satisfies the conditions outlined in section 3.2.2, then equations (3.18) and (3.19) are both necessary and sufficient.

3.4 GENERAL CONTINUOUS REVIEW

Let us now consider the continuous review problem stated in section 2.2.2 which was:

$$\text{minimize} \quad Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx \ ,$$

subject to:

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} - \mu_i) \leq K_1 ,$$

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \leq K_2 ,$$

$$Q_i \geq 0, \text{ and } r_i \text{ unrestricted} .$$

3.4.1 Necessary Conditions

As a result of condition (1) in section 3.1, we have the following equations for all i

$$-\frac{\lambda_i}{Q_i} F_i^c(r_i) - \eta_1 c_i = 0 \tag{3.20}$$

and

$$-\frac{\lambda_i}{Q_i{}^2} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)dx - \frac{\eta_1 c_i}{2} + \frac{\eta_2 \lambda_i}{Q_i{}^2} \geq 0 . \tag{3.21}$$

Also, because of condition (1), we know $Y^o F_Y(X^o, Y^o, \P) = 0$. If we modify equation (3.21) by multiplying through by Q_i, we obtain

$$-\frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)dx - \frac{\eta_1 c_i Q_i}{2} + \frac{\eta_2 \lambda_i}{Q_i} = 0 . \tag{3.22}$$

As a result of condition (2) we have

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} - \mu_i) - K_1 \leq 0 \tag{3.23}$$

and

$$\eta_1 \left[\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} - \mu_i) \right] = 0. \tag{3.24}$$

Equation (3.20) can be solved for η_1 which yields

$$\eta_1 = -\frac{\lambda_i}{c_i Q_i} F_i^c(r_i) \quad . \tag{3.25}$$

Since $\lambda_i > 0$, $c_i > 0$ and $Q_i \geq 0$, η_1 is always less than zero unless r_i exceeds the greatest possible lead time demand. Assuming this is not the case, equations (3.23) and (3.24) reduce to

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} - \mu_i) - K_1 = 0 \ . \tag{3.26}$$

In addition, from condition (2) we have

$$\sum_{i=1}^{n} \frac{\lambda_i}{c_i} - K_2 \leq 0$$

and

$$\eta_2\left[\sum_{i=1}^{n} \frac{\lambda_i}{c_i} - K_2\right] = 0 \quad .$$

Then considering equation (3.22) and substituting for η_1 equation (3.25), we have

$$-\frac{\lambda_i}{Q_i}\int_{r_i}^{\infty} (x-r_i)f_i(x,\tau_i)dx + \frac{\lambda_i}{2} F_i^c(r_i) + \frac{\eta_2\lambda_i}{Q_i} = 0 \ .$$

Solving for η_2 yields

$$\eta_2 = -\frac{Q_i}{2} F_i^c(r_i) + \int_{r_i}^{\infty} (x-r_i)f_i(x,\tau_i)dx$$

or

$$\eta_2 = -(r_i + \frac{Q_i}{2})F_i^c(R_i) + \int_{r_i}^{\infty} xf_i(x,\tau_i)dx \quad . \tag{3.27}$$

The right-hand side of equation (3.27) is not always negative. However, theorem 1 requires $\eta_i \leq 0$. This suggests two possible cases for consideration.

<u>Case I.</u> $\eta_2 < 0$, $\eta_1 < 0$

As a result of condition (2), the necessary conditions in this care are:

$$-\frac{\lambda_i}{Q_i} F_i^c(r_i) - \eta_1 c_i = 0 \ , \tag{3.20}$$

$$- \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx - \frac{\eta_1 c_i Q_i}{2} + \frac{\eta_2 \lambda_i}{Q_i} = 0 \ , \qquad (3.22)$$

$$\sum_{i=1}^{n} c_i (r_i + \frac{Q_i}{2} - \mu_i) - K_1 = 0 \ , \qquad (3\text{-}26)$$

and

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} - K_2 = 0 \ . \qquad (3.28)$$

<u>Case II</u>. $\eta_2 = 0, \ \eta_1 < 0$

The necessary conditions in this case are

$$- \frac{\lambda_i}{Q_i} F_i^c(r_i) - \eta_1 c_i = 0 \ , \qquad (3.20)$$

$$- \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx - \frac{\eta_1 c_i Q_i}{2} = 0 \ , \qquad (3.27)$$

and

$$\sum_{i=1}^{n} c_i (r_i + \frac{Q_i}{2} - \mu_i) = K_1 \ . \qquad (3.25)$$

Since the equations above cannot be solved for r^o and Q^o in general, some iterative process or search technique seems desirable. One such technique was used in solving an example problem stated in section 5.2.

Beginning with the solution to the simplified continuous review problem, which would be a feasible one, compute the value of η_2 for all i. Find

$$h = \min[\eta_2(i)]$$

and

$$f = \max[\eta_2(i)] \ .$$

It is now desirable to change the feasible solution obtained previously so that h increases and f decreases until h = f. When this occurs, we have satisfied the necessary conditions for optimality. If $\eta_2(j) = h$ and $\eta_2(k) = f$, we reduce Q_j by 1 and increase Q_k using equation (3.28) such that (3.28) is still satisfied. Then we solve another simplified continuous review problem using the new values of Q_j and Q_k. We continue until h = f.

The algorithm described above is an elementary approach to the problem. More sophisticated approaches are available. Among these is a search technique proposed by Fiacco and McCormick[3]. This search technique does not require a convex objective function or a convex constraint region. Of course, any solution is strictly a local minimum in this case. The Fiacco - McCormick search is limited by the requirement to compute the inverse of a (2n + 2) x (2n + 2) matrix. This limits the size of the inventory for which this technique is useful.

3.4.2 Sufficient Conditions

If the objective function was convex we could apply the principle used in the preceding models to determine sufficiency conditions. However, Veinott [4] states that the objective function, equation (2.8), in general is not convex. Veinott also discusses the conditions under which it is convex. The following sufficiency conditions do not depend upon convexity of the objective function.

Consider the problem

$$\min f(X,Y)$$

subject to:

$$g_i(X,Y) \leq 0 \quad i=1,\cdots,m$$

The set of m equations $g_i(X,Y) \leq 0$ can be written as a vector equation

$$G(X,Y) \leq 0 \quad .$$

Partition $G(X,Y) \leq 0$ such that

$$G(X^o,Y^o) = [G^{(1)}(X^o,Y^o),G^{(2)}(X^o,Y^o)]$$

where

$$G^{(1)}(X^o,Y^o) = 0$$

and

$$G^{(2)}(X^o,Y^o) < 0 \ ,$$

and either may be empty.

Partition $G^{(1)}(X^o,Y^o)$ as

$$G^{(1)}(X^o,Y^o) = [G^*(X^o,Y^o),G^{**}(X^o,Y^o)]$$

where the η_i associated with each element of $G^{**}(X^o,Y^o)$ is zero and the η_i associated with each element of $G^*(X^o,Y^o)$ is different from zero.

Define D as the 2n x m matrix

$$D = \left[\frac{\partial g_i}{\partial x_i} , \frac{\partial g_i}{\partial y_i} \right] .$$

Partition D^o and $¶^o$ as

$$D^o = [D^*,D^{**}] \ ,$$

$$¶^o = \begin{bmatrix} ¶^* \\ 0 \end{bmatrix} \quad .$$

If we define $F(x,y,¶)$ as

$$F(x,y,¶) = f(x,y) - G^*(x,y)¶$$

and define $[E_F^o]$ as

$$[E_F^o] = \begin{bmatrix} \dfrac{\partial^2 F}{\partial x_i \partial x_j} , & \dfrac{\partial^2 F}{\partial x_i \partial y_j} \\ \\ \dfrac{\partial^2 F}{\partial y_j \partial x_i} , & \dfrac{\partial^2 F}{\partial y_j \partial y_i} \end{bmatrix} .$$

King [5] states and proves the following theorem relating to sufficient conditions.

<u>Theorem 2</u>. In order that f (X,Y) have a local minimum at (X,Y), it is sufficient that (in addition to Theorem 1)

$$(X-X^o, Y-Y^o)^T [E_F^o] (X-X^o, Y-Y^o) > 0$$

for all X and Y satisfying

$$(X-X^o, Y-Y^o) D^* = 0 .$$

It appears that Theorem 2 provides sufficient conditions for Case I and Case II. The only difference being, in Case I, that

$G^*(X^o, Y^o)$ includes $g_1(X^o, Y^o)$ and $g_2(X^o, Y^o)$.

In Case II, $G^*(X^o, Y^o)$ only includes $g_1(X^o, Y^o)$.

While Theorem 2 is interesting from a theoretical standpoint, it does not provide a direct computational verification of sufficiency for any particular solution. However, in Section 5.2 such a procedure is indicated.

CHAPTER 4

EXTENSIONS OF THE GENERAL PROBLEM

4.1 A PROCUREMENT BUDGET CONSTRAINT, CONTINUOUS REVIEW

Suppose instead of a limit on the average investment, we are given a limit on the amount of money we may obligate for procurement in a period of time. This constraint can be stated as

$$\sum_{i=1}^{n} c_i n_i Q_i \leq B \tag{4.1}$$

where n_i is the number of orders for the ith item per unit time.

If we consider a continuous review system with back-orders allowed, the value of n_i for the ith item is the non-negative integer which satisfies

$$r_i < a_i - x_i + n_i Q_i \leq r_i + Q_i \tag{4.2}$$

where a_i = the asset position of the ith item at the beginning of the period, and

x_i = the demand random variable per unit time.

Assuming the inventory position at the end of the period is $r_i + \frac{Q_i}{2}$ and relaxing the requirement that n be integer valued, then equation (4.2) becomes

$$a_i - x_i + n_i Q_i = r_i + \frac{Q_i}{2} \quad . \tag{4.3}$$

Solving for n_i,

$$n_i Q_i = r_i + \frac{Q_i}{2} + x_i - a_i$$

and

$$n_i = \frac{r_i + \frac{Q_i}{2} + x_i - a_i}{Q_i} \tag{4.4}$$

By substituting (4.4) into (4.1) we have

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2} + x_i - a_i) \leq B . \qquad (4.5)$$

By taking the expected value of (4.5), the random variable x_i becomes the parameter λ_i or expected annual demand. This results in the following equation:

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2}) + \sum_{i=1}^{n} c_i\lambda_i - \sum_{i=1}^{n} a_i c_i = B . \qquad (4.6)$$

This constraint can be rewritten as

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2}) \leq B - \sum_{i=1}^{n} c_i\lambda_i + \sum_{i=1}^{n} a_i c_i = B' \qquad (4.7)$$

or

$$\sum_{i=1}^{n} c_i(r_i + \frac{Q_i}{2}) \leq B' . \qquad (4.8)$$

Notice the constraint (4.8) is of the same form as equation (2.6), the first constraint associated with the general continuous review model. From this analysis we conclude a constraint on obligation authority presents a problem similar to the general continuous review problem.

An example of a system operating under a procurement budget is the Navy UICP program discussed in the introduction. If we desire to minimize shortages, the model is

$$\min \quad Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx$$

subject to:

$$\sum_{i=1}^{n} \frac{\lambda_i}{Q_i} \leq K_2$$

$$\sum_{i=1}^{n} c_i (r_i + \frac{Q_i}{2}) \leq B' \quad .$$

4.2 A WEIGHTED SHORTAGES FORMULATION

The basic formulation of the problem may be extended to several situations. If we let d_i represent the cost of a shortage of the ith item, the objective can be stated as

$$\min \quad Z = \sum_{i=1}^{n} \frac{\lambda_i}{Q_i} d_i \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx \; .$$

The addition of the weighting factor d_i does not increase the difficulty of solving the general problem.

4.3 MINIMIZE TIME WEIGHTED SHORTAGES

Recently in the Naval Supply Systems Command, the number one objective of Naval logistics management was revised to be the minimization of time weighted shortages. In other words, they desire to minimize the amount of time a customer must wait to receive his material. If we assume the distribution of lead time demand for the ith item is normal (μ_i, σ_i) and a continuous review system, the objective function can be stated, Hadley and Whitin [1], as

$$\min \sum_{i=1}^{n} B(Q,r)$$

where
$$B(Q,r) = \frac{1}{Q}[\beta(r) - \beta(r+Q)]$$

and
$$\beta(r) = \frac{1}{2}[\sigma^2 + (r-\mu)^2] F^c(r) - \frac{\sigma}{2}(r-\mu) f(r) \; .$$

We have formulated the problem since this problem is one of primary interest to the Navy today. However, no attempt was made to provide a solution procedure or to determine necessary and sufficient conditions for optimality.

CHAPTER 5

SOME EXAMPLE PROBLEM SOLUTIONS

5.1 AN EXAMPLE OF THE SIMPLIFIED CONTINUOUS REVIEW MODEL

Let us consider an inventory of three items. The solution technique for n items would be similar. We assume the distribution of lead time demand is normal with mean μ_i and variance σ_i for the ith item. These items have the following characteristics.

Parameter	Item 1	Item 2	Item 3
λ_i	1000	1500	2000
c_i	1	10	20
μ_i	100	200	300
σ_i	100	100	200

$$K_1 = \$8{,}000 \qquad K_2 = 15 \quad .$$

The problem is then

$$\min \quad Z = \sum_{i=1}^{3} K\sqrt{c_i \lambda_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)\,dx$$

subject to:

$$\sum_{i=1}^{3} c_i\left(r_i + \frac{Q_i}{2}\right) \leq 8000 + \sum_{i=1}^{3} c_i\mu_i \quad .$$

In section 3.2, equation (3.6) determines the value of the constant K which, in turn, fixes the order quantities. Initially, then, the value of K is determined by

$$K = \frac{K_2}{\sum_{i=1}^{n} \sqrt{c_i \lambda_i}} = .04244 \quad .$$

Utilizing equation (3.4), $Q_k = \frac{1}{K} \sqrt{\frac{\lambda_i}{c_i}}$, the order quantities can be determined as

$$Q_1 = 746 \ ,$$

$$Q_2 = 289 \ , \quad \text{and}$$

$$Q_3 = 236 \ .$$

Since the Q_i have been determined, the problem now can be written as

$$\min \quad Z = \sum_{i=1}^{3} .04244\sqrt{c_i \lambda_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx$$

subject to:

$$\sum_{i=1}^{3} c_i r_i \leq 8000 + \sum_{i=1}^{3} c_i \mu_i - \sum_{i=1}^{3} \frac{Q_i c_i}{2} = 11{,}922 \ .$$

From equation (3.12) we see the solution to the above problem is that vector r^o such that

$$F_i^c(r_i) = -\frac{\eta c_i}{K\sqrt{\lambda_i c_i}} \ , \text{ for all } i \ . \tag{3.12}$$

For each value of η there will be some vector r for which equation (3.12) is satisfied. However, since the objective function is convex, there exists only one vector r such that equation (3.12) and the constraint (3.8) are satisfied simultaneously. As we decrease η from zero, we see that at $\eta = -.102$,

$$F_1^c(r_1) = .0760 \ ,$$

$$F_2^c(r_2) = .1965 \text{ , and}$$

$$F_3^c(r_3) = .2410 \text{ ,}$$

which implies,

$$r_1 = 243.30 \text{ ,}$$

$$r_2 = 285.40 \text{ , and}$$

$$r_3 = 440.80 \text{ .}$$

Checking the constraint, we find

$$\sum_{i=1}^{3} c_i r_i = 11923.30 \text{ ,}$$

which is within two dollars of the required average investment limit.

The order quantities and reorder points determined yield an expected number of shortages per unit time for each item. The expression for the expected number of shortages per item per unit time is

$$Z_i = \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx \text{ .} \tag{5.1}$$

When the distribution of lead time demand is normal, (5.1) can be written, Hadley and Whitin [1], as

$$Z_i = \frac{\lambda_i}{Q_i} \left[(\mu_i - r_i) \Phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) + \sigma_i \phi\left(\frac{r_i - \mu_i}{\sigma_i}\right) \right] \tag{5.2}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^{\infty} \exp\left(-\frac{x^2}{2}\right) dx \text{ and } \phi(z) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{z^2}{2}\right) \text{ .}$$

Using (5.2), the expected shortages per unit time for each item is computed as

$$Z_1 = \frac{1000}{746} [-(143.30)(.0760) + 100(.1428)] = 4.52 ,$$

$$Z_2 = \frac{1500}{289} [-(85.40)(.1965) + 100(.2770)] = 56.7 , \text{ and}$$

$$Z_3 = \frac{2000}{236} [-(140.8)(.2410) = 200(.3114)] = 240.0 .$$

The total expected shortages per unit time for the inventory under consideration is

$$Z = \sum_{i=1}^{3} Z_i = 4.52 + 56.70 + 240.00 = 301.22 .$$

5.2 GENERAL CONTINUOUS REVIEW EXAMPLE

Once again let us consider the inventory of 3 items from 5.1. We shall again assume the distribution of lead time demand is normal (μ_i, σ_i^2) for the ith item. The items under consideration have the following characteristics.

Parameter	Item 1	Item 2	Item 3
λ_i	1000	1500	2000
c_i	1	10	20
μ_i	100	200	300
σ_i	100	100	200

$K_1 = \$8,000 \qquad K_2 = 15 .$

Reviewing section (3.4) we note that the solution (r^o, Q^o) must satisfy

$$-\frac{\lambda_i}{Q_i} F_i^c(r_i) = \eta_1 c_i , \tag{3.18}$$

$$-\frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx - \frac{\eta_1 c_i}{2} + \frac{\eta_2 \lambda_i}{Q_i} = 0 , \tag{3.20}$$

$$\sum_{i=1}^{3} c_i(r_i + \frac{Q_i}{2} - \mu_i) = K_1, \tag{3.23}$$

and

$$\sum_{i=1}^{3} \frac{\lambda_i}{Q_i} = K_2 \quad . \tag{3.24}$$

Equation (3.18) is now rewritten as

$$\eta_1 = - \frac{\lambda_i}{c_i Q_i} F_i^C(r_i), \text{ for all } i \; . \tag{5.3}$$

Substituting (5.3) for η_1 in (3.20), we get

$$\eta_2 = - \frac{1}{2} F_i^C(r_i) + \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i)dx \; . \tag{5.4}$$

In other words, the necessary conditions for solution are present when

$$\eta_1 = - \frac{\lambda_i}{c_i Q_i} F_i^C(r_i) = - \frac{\lambda_i}{c_j Q_j} F_j^C(r_j),$$

for all i and j, and

$$\eta_2 = \int_{r_i}^{\infty} x f_i(x,\tau_i)dx - (\frac{1}{2} + r_i)F_i^C(r_i)$$

$$= \int_{r_j}^{\infty} x f_j(x,\tau_j)dx - (\frac{1}{2} + r_j)F_j^C(r_j) \quad ,$$

for all i and j. In addition, both constraint equations (3.26) and (3.28) must be satisfied.

Using the search routine described in section (3.4.1), this example was solved with the following results.

Variable	1	2	3
r	269.65	307.11	409.79
Q	483.08	229.12	313.31
η_1	-.0930	-.0930	-.0930
η_2	-9.013	-9.012	-9.012

The value of the objective function (Z) was 285.20. Since $\eta_1 < 0$ and $\eta_2 < 0$, Case I applied and Case II was not applicable. If η_2 had been non-negative, we would have used the conditions of Case II for solution.

Next we must determine whether the solution obtained above satisfies the sufficiency conditions stated in theorem 2 in section 3.4.2 for a relative min. That is we must determine if

$$h^T E_F h > 0 ,$$

for

$$hD^* = 0 .$$

E_F, h and D^* are defined as in section 3.4.2.

The theorem stated below is indicated by Hadley [6] and proved in Hancock [7].

Consider the matrix A

$$A = \begin{bmatrix} E_F - \lambda I & D^{*T} \\ D^* & 0 \end{bmatrix} .$$

Theorem 3. If the roots of $||A||$ are all positive then f(X,Y) takes on a strong relative minimum at (X^o, Y^o).

The procedure for applying theorem 3 is simple enough. Construct the matrix A. Take the determinant of A and set it equal to zero. The result will be an (2n x 2) degree polynomial in λ. Using Decartes rule of signs, the number of positive roots of $||A||$ can be determined. For example, suppose $||A|| = \lambda^2 + 2\lambda + 1 = 0$. Here we have no sign changes. According to Decartes rule of signs there are no positive roots. However in the equation $\lambda^2 - 2\lambda + 1 = 0$ there are 2 sign changes which indicates 2 positive roots.

To show sufficiency of the solution states previously we form the matrix A evaluated at (Q^o, r^o) which is

$$A = \begin{bmatrix} .196-\lambda & 0 & 0 & .000193 & 0 & 0 & 1 & 0 \\ 0 & 1.475-\lambda & 0 & 0 & .00405 & 0 & 10 & 0 \\ 0 & 0 & 2.19-\lambda & 0 & 0 & .006 & 20 & 0 \\ .000193 & 0 & 0 & -.000126 & 0 & 0 & .5 & .00428 \\ 0 & .00405 & 0 & 0 & -.000696 & 0 & 5 & .0286 \\ 0 & 0 & .006 & 0 & 0 & .0018 & 10 & .0205 \\ 1 & 10 & 20 & .5 & 5 & 10 & 0 & 0 \\ 0 & 0 & 0 & .00428 & .0286 & .0205 & 0 & 0 \end{bmatrix}$$

Taking the determinant of A and setting it equal to zero we find all of the roots are positive indicating the solution (Q^o, r^o) is a strong relative minimum.

5.3 SIMPLIFIED CONTINUOUS REVIEW WITH A PROCUREMENT BUDGET CONSTRAINT

In this section we shall consider a three item inventory and again assume the distribution of lead time demand for the ith item is normal (μ_i, σ_i). The items under consideration have the following characteristics

Parameter	1	2	3
λ_i	1000	1500	2000
c_i	1	10	20
μ_i	100	200	300
σ_i	100	100	200
a_i	100	200	300

$B = \$64,000 \qquad K_2 = 15$

The general problem is

$$\min \quad Z = \sum_{i=1}^{3} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx$$

subject to:

$$\sum_{i=1}^{3} c_i(r_i + \frac{Q_i}{2} + \lambda_i - a_i) \leq B$$

$$\sum_{i=1}^{3} \frac{\lambda_i}{Q_i} \leq K_2 .$$

Since λ_i, a_i, and c_i are known constants, the first constraint can be rewritten as

$$\sum_{i=1}^{3} c_i(r_i + \frac{Q_i}{2}) \leq 64{,}000 - \sum_{i=1}^{3} \lambda_i c_i + \sum_{i=1}^{3} a_i c_i, \text{ or}$$

$$\sum_{i=1}^{3} c_i(r_i + \frac{Q_i}{2}) \leq 64{,}000 - 56{,}000 + 8100 = 16{,}100. \tag{5.5}$$

If we also assume

$$Q_i = \frac{1}{K} \sqrt{\frac{\lambda_i}{c_i}}$$

as was done in section 3.2 and section 5.1, equation (5.5) reduces to

$$\sum_{i=1}^{n} c_i r_i \leq 16{,}100 - \sum_{i=1}^{n} c_i \frac{Q_i}{2} , \tag{5.6}$$

or

$$\sum_{i=1}^{n} c_i r_i \leq 11922 .$$

The problem can now be stated as

$$\min \quad Z = \sum_{i=1}^{3} \frac{\lambda_i}{Q_i} \int_{r_i}^{\infty} (x-r_i) f_i(x,\tau_i) dx$$

subject to:

$$\sum_{i=1}^{n} c_i r_i \leq 11{,}922 \quad .$$

Observe, however, that this problem was solved in section 5.1.

CHAPTER 6

SUMMARY AND CONCLUSIONS

It is apparent that the models proposed do not represent the ultimate answer in multi-item inventory theory. The simplified continuous review problem stated in section 3.2 represents a model of the Navy's VOSL program. From the example problems we see the general problem provides a solution which is better than the solution of the simplified continuous review problem. The major advantage of the simplified continuous review model is its computational ease.

While the solution to the general problem as stated is theoretically correct, an efficient algorithm for rapid location of stationary points and subsequent verification of sufficiency conditions is at present not available. There is certainly room for future research on this topic.

The assumption that the expected number of orders which can be processed is a well known constant is perhaps optimistic. However, if we look at the general problem as a two criterion problem, we can generate the following efficiency curve. One method of generating this curve would be to

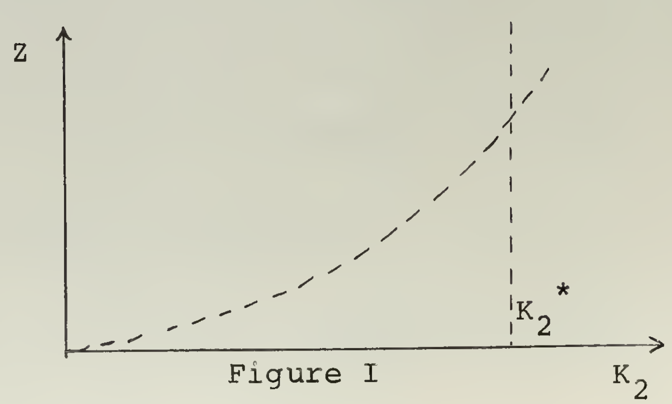

Figure I

solve the general problem for several values of the parameter K_2 (the expected number of orders). If these values of K_2 versus the expected shortages per unit time are plotted, the curve represented by Figure I is generated.

It is apparent that for every organization of N people, there is some maximum number of orders which can be processed per unit time. This point is represented by K_2^*. However, we contend that each individual manager must examine the alternatives represented by the efficiency curve in Figure I and select that point at which he desires to operate.

Another approach to the problem would be to formulate a vector minimization problem. For example,

$$\min \quad [Z, K_2] \ ,$$

subject to:

$$\sum_{i=1}^{3} c_i (r_i + \frac{Q_i}{2} - \mu_i) \leq K_1 \ .$$

Hadley [6] indicates a procedure for solving problems of this type. Such an approach to the problem proposed represents a fertile area for future work.

From the example problems given, it appears the Navy's method of fixing Q and solving for r_i (as we did in the simplified formulation of section 3.2) does not result in an optimal solution to the problem of minimizing shortages per unit time. The use of the unknown parameters, holding cost, shortage cost, and order cost presents a possible source of error.

We contend the constraints on average investment and the expected number of orders actually imply the values of the unknown parameters. Since the values of the constraints are more easily determined than the order cost and holding cost, the models proposed seem much more appropriate than the traditional variable cost minimization models.

While an efficient algorithm for solution of the general problem has not been presented, the advent of high speed computers has opened the field of iterative solution procedures. It should be only a matter of time until a procedure is available which can be reasonably applied to a large inventory. However, the techniques discussed in section 3.4.1 are feasible only for small inventories or subsets of the larger inventory. For instance the problem could be solved for the entire inventory using the simplified continuous review model. As stated previously, the simplified continuous review algorithm is computationally feasible for large inventories. The inventory manager could then select subsets of items whose decision variables intuitively appear to be unreasonable. We can then formulate and solve the general model for the subset of items using the budget allocated, to those items by the simplified continuous review model as K_1. The number of orders allocated to the subset of items would become the constant K_2. Selection of a subset of the n item inventory is necessary since the general solution algorithms mentioned are feasible only if the number of items is small. We can guarantee that the value of the objective function will at worst be the same as the simplified continuous review model and in all likelihood, it will decrease.

While the multi-item problem has been solved when funds are unlimited, the assumption must be made that there are no interactions among items (i.e., the problem degenerates into N single item problems). For instance, it must be assumed that enough materials handling equipment is available to handle all material, enough warehouse space is available and numerous other possible interactions do not exist. Each of these interactions, including a limitation on funds, represents a constraint on some resource within the system. It seems logical, then, that the next step in the formulation of multi-item inventory problems should be of the form presented in this study.

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13. ABSTRACT

Historically multi-item inventory control has been modeled by assuming that each item can be treated independently in a variable cost minimization formulation. In this paper independence between items is not assumed. Constraints on total system operating characteristics create inter-item dependencies. Optimal policies are determined from a goal-constraint formulation. This is done without reliance upon unknown parameters such as order cost and carrying cost which the traditional theory leans on heavily. A group of models are presented, with necessary and sufficient conditions for optimal solutions provided for each. In addition, solution algorithms are indicated for the major models. An algorithm for verification of sufficiency conditions for a non-convex objective function is also provided.

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